

# EQUIVARIANT EXTENSION PROPERTIES OF COSET SPACES OF LOCALLY COMPACT GROUPS AND APPROXIMATE SLICES

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**ABSTRACT.** We prove that for a compact subgroup  $H$  of a locally compact Hausdorff group  $G$ , the following properties are mutually equivalent: (1)  $G/H$  is a manifold, (2)  $G/H$  is finite-dimensional and locally connected, (3)  $G/H$  is locally contractible, (4)  $G/H$  is an ANE for paracompact spaces, (5)  $G/H$  is a metrizable  $G$ -ANE for paracompact proper  $G$ -spaces having a paracompact orbit space. A new version of the Approximate slice theorem is also proven in the light of these results.

## 1. INTRODUCTION

In the fundamental work of R. Palais [30], it was established that for any compact Lie group  $G$  and its compact subgroup  $H$  the coset space  $G/H$  has the following equivariant extension property: for every normal  $G$ -space  $X$  and a closed invariant subset  $A \subset X$ , every  $G$ -map  $f : A \rightarrow G/H$  extends to a  $G$ -map  $f' : U \rightarrow G/H$  defined on an invariant neighborhood  $U$  of  $A$ . In this case one writes  $G/H \in G\text{-ANE}$ . This property of  $G$  is equivalent to the, so-called, Exact slice theorem: every orbit in a completely regular  $G$ -space  $X$  is a neighborhood  $G$ -equivariant retract of  $X$  (see [30]).

In general, when  $G$  is a compact non-Lie group, the Exact slice theorem is no longer true (see [7]). At the same time, among the coset spaces  $G/H$  of a compact non-Lie group  $G$ , still there are many which possess the property  $G/H \in G\text{-ANE}$ . This observation led the author in [7] to the, so-called, Approximate slice theorem, which is valid for every compact group of transformations. It claims that given a point  $x$  and its neighborhood  $O$  in a  $G$ -space, there exists a  $G$ -invariant neighborhood  $U$  of  $x$  that admits a  $G$ -map  $f : U \rightarrow G/H$  to a coset space  $G/H$  with a compact subgroup  $H \subset G$  such that  $G/H \in G\text{-ANE}$  and  $x \in f^{-1}(eH) \subset O$ .

On the other hand, in 1961 the Exact slice theorem was extended by R. Palais [31] to the case of proper actions of non-compact Lie groups. Validity of the property  $G/H \in G\text{-ANE}$  for every compact subgroup  $H$  of a noncompact Lie group  $G$  follows from Palais' Exact slice theorem; this was proved later in E. Elfving [18, p. 23-24].

It is one of the purposes of this paper to prove (see Proposition 4.11 and Theorem 4.14) that if  $G$  is a locally compact group and  $H$  a compact subgroup of  $G$  then the following properties are mutually equivalent: (1)  $G/H$  is a manifold, (2)  $G/H$  is finite-dimensional

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and locally connected, (3)  $G/H$  is locally contractible, (4)  $G/H$  is an ANE for paracompact spaces, (5)  $G/H$  is a metrizable  $G$ -ANE for paracompact proper  $G$ -spaces having a paracompact orbit space.

One should clarify that in the case of noncompact group actions the property  $G/H \in G\text{-ANE}$  reads as follows: for every paracompact proper  $G$ -space  $X$  having a paracompact orbit space, every  $G$ -map  $f : A \rightarrow G/H$  from a closed invariant subset  $A \subset X$  extends to a  $G$ -map  $f' : U \rightarrow G/H$  over an invariant neighborhood  $U$  of  $A$ .

The equivalence of the above properties (3) and (5) is claimed also in [10, Proposition 3.2] However, the proof given in [10] is valid only for locally compact almost connected groups. Proposition 4.11 and Theorem 4.14, in particular, fill up this gap also.

As in the compact case [7], the equivariant extension properties of coset spaces are conjugated with approximate slices. In section 5 we prove a new version of the Approximate slice theorem valid for proper actions of arbitrary locally compact groups. Section 3 plays an auxiliary role. Here we establish a special equivariant embedding of a coset space  $G/H$  into a  $G\text{-AE}(\mathcal{P})$ -space (see Proposition 3.7), which is further used in the proof of Theorem 4.14.

## 2. PRELIMINARIES

Throughout the paper the letter  $G$  will denote a locally compact Hausdorff group unless otherwise stated; by  $e$  we shall denote the unity of  $G$ .

All topological spaces and topological groups are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of  $G$ -spaces or topological transformation groups can be found in G. Bredon [15] and in R. Palais [30]. Our basic reference on proper group actions is Palais' article [31]. Other good sources are [1], [27]. For equivariant theory of retracts the reader can see, for instance, [4], [5] and [9].

For the convenience of the reader, we recall however some more special definitions and facts below.

By a  $G$ -space we mean a triple  $(G, X, \alpha)$ , where  $X$  is a topological space, and  $\alpha : G \times X \rightarrow X$  is a continuous action of the group  $G$  on  $X$ . If  $Y$  is another  $G$ -space, a continuous map  $f : X \rightarrow Y$  is called a  $G$ -map or an equivariant map if  $f(gx) = gf(x)$  for every  $x \in X$  and  $g \in G$ . If  $G$  acts trivially on  $Y$  then we will use the term "invariant map" instead of "equivariant map".

By a normed linear  $G$ -space (resp., a Banach  $G$ -space) we shall mean a  $G$ -space  $L$ , where  $L$  is a normed linear space (resp., a Banach space) on which  $G$  acts by means of *linear isometries*, i.e.,  $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$  and  $\|gx\| = \|x\|$  for all  $g \in G$ ,  $x, y \in L$  and  $\lambda, \mu \in \mathbb{R}$ .

If  $X$  is a  $G$ -space then for a subset  $S \subset X$ ,  $G(S)$  denotes the saturation of  $S$ ; i.e.,  $G(S) = \{gs \mid g \in G, s \in S\}$ . In particular,  $G(x)$  denotes the  $G$ -orbit  $\{gx \in X \mid g \in G\}$  of  $x$ . If  $G(S) = S$  then  $S$  is said to be an invariant or  $G$ -invariant set. The orbit space is denoted by  $X/G$ .

For a closed subgroup  $H \subset G$ , by  $G/H$  we will denote the  $G$ -space of cosets  $\{gH \mid g \in G\}$  under the action induced by left translations.

If  $X$  is a  $G$ -space and  $H$  a closed normal subgroup of  $G$  then the  $H$ -orbit space  $X/H$  will always be regarded as a  $G/H$ -space endowed with the following action of the group  $G/H$ :  $(gH) * H(x) = H(gx)$ , where  $gH \in G/H$ ,  $H(x) \in X/H$ .

For any  $x \in X$ , the subgroup  $G_x = \{g \in G \mid gx = x\}$  is called the stabilizer (or stationary subgroup) at  $x$ .

A compatible metric  $\rho$  on a  $G$ -space  $X$  is called invariant or  $G$ -invariant if  $\rho(gx, gy) = \rho(x, y)$  for all  $g \in G$  and  $x, y \in X$ .

A locally compact group  $G$  is called *almost connected* whenever its quotient space of connected components is compact.

Let  $X$  be a  $G$ -space. Two subsets  $U$  and  $V$  in  $X$  are called *thin* relative to each other [31, Definition 1.1.1] if the set

$$\langle U, V \rangle = \{g \in G \mid gU \cap V \neq \emptyset\},$$

called *the transporter* from  $U$  to  $V$ , has a compact closure in  $G$ .

A subset  $U$  of a  $G$ -space  $X$  is called *small* if every point in  $X$  has a neighborhood thin relative to  $U$ . A  $G$ -space  $X$  is called *proper* (in the sense of Palais) if every point in  $X$  has a small neighborhood.

Each orbit in a proper  $G$ -space is closed, and each stabilizer is compact [31, Proposition 1.1.4]. Furthermore, if  $X$  is a compact proper  $G$ -space, then  $G$  has to be compact as well.

Important examples of proper  $G$ -spaces are the coset spaces  $G/H$  with  $H$  a compact subgroup of a locally compact group  $G$ . Other interesting examples the reader can find in [1], [8], [13], [26] and [27].

In the present paper we are especially interested in the class  $G\mathcal{P}$  of all paracompact proper  $G$ -spaces  $X$  that have paracompact orbit space  $X/G$ . It is a long time standing open problem whether the orbit space of any paracompact proper  $G$ -space is paracompact (see [22] and [1]).

A  $G$ -space  $Y$  is called an equivariant neighborhood extensor for a given  $G$ -space  $X$  (notation:  $Y \in G\text{-ANE}(X)$ ) if for any closed invariant subset  $A \subset X$  and any  $G$ -map  $f : A \rightarrow Y$ , there exist an invariant neighborhood  $U$  of  $A$  in  $X$  and a  $G$ -map  $\psi : U \rightarrow Y$  such that  $\psi|_A = f$ . If in addition one always can take  $U = X$ , then we say that  $Y$  is an

equivariant extensor for  $X$  (notation:  $Y \in G\text{-AE}(X)$ ). The map  $\psi$  is called a  $G$ -extension of  $f$ .

A  $G$ -space  $Y$  is called an equivariant absolute neighborhood extensor for the class  $G\text{-}\mathcal{P}$  (notation:  $Y \in G\text{-ANE}(\mathcal{P})$ ) if  $Y \in G\text{-ANE}(X)$  for any  $G$ -space  $X \in G\text{-}\mathcal{P}$ . Similarly, if  $Y \in G\text{-AE}(X)$  for any  $X \in G\text{-}\mathcal{P}$ , then  $Y$  is called an equivariant absolute extensor for the class  $G\text{-}\mathcal{P}$  (notation:  $Y \in G\text{-AE}(\mathcal{P})$ ).

**Theorem 2.1** (Abels [1]). *Let  $G$  be a locally compact group and  $T$  a  $G$ -space such that  $T \in K\text{-AE}(\mathcal{P})$  for every compact subgroup  $K \subset G$ . Then  $T \in G\text{-AE}(\mathcal{P})$ .*

**Remark 2.2.** *In [1, Theorem 4.4] the result is stated only for  $G\text{-AE}(\mathcal{M})$  while the proof is valid also for  $G\text{-AE}(\mathcal{P})$ , where  $G\text{-}\mathcal{M}$  stands for the class of all proper  $G$ -spaces that are metrizable by a  $G$ -invariant metric.*

**Corollary 2.3.** *Let  $G$  be a locally compact group and  $B$  a Banach  $G$ -space. Then  $B \in G\text{-AE}(\mathcal{P})$ .*

*Proof.* By a result of E. Michael [28],  $B$  is an  $\text{AE}(\mathcal{P})$ . Then, it follows from [3, Main Theorem] that  $B$  is a  $K\text{-AE}(\mathcal{P})$  for every compact subgroup  $K$  of  $G$ . It remains to apply Theorem 2.1.  $\square$

Let us recall the well-known definition of a slice [31, p. 305]:

**Definition 2.4.** *Let  $X$  be a  $G$ -space and  $H$  be a closed subgroup of  $G$ . An  $H$ -invariant subset  $S \subset X$  is called an  $H$ -slice in  $X$  if  $G(S)$  is open in  $X$  and there exists a  $G$ -equivariant map  $f : G(S) \rightarrow G/H$  such that  $S = f^{-1}(eH)$ . The saturation  $G(S)$  is called a tubular set. If  $G(S) = X$  then we say that  $S$  is a global  $H$ -slice of  $X$ .*

The following result of R. Palais [31, Proposition 2.3.1] plays a central role in the theory of topological transformation groups:

**Theorem 2.5** (Exact slice theorem). *Let  $G$  be a Lie group,  $X$  a proper  $G$ -space and  $a \in X$ . Then there exists a  $G_a$ -slice  $S \subset X$  such that  $a \in S$ .*

### 3. EQUIVARIANT EMBEDDINGS INTO A $G\text{-AE}(\mathcal{P})$ -SPACE

Recall that the letter  $G$  always denotes a locally compact Hausdorff group.

The main result of this section is Proposition 3.7 which provides a special equivariant embedding of a coset space  $G/H$  into a  $G\text{-AE}(\mathcal{P})$ -space; it is used in the proof of Theorem 4.14 below.

We begin with the following lemma proved in [2, Lemma 2.3]:

**Lemma 3.1.** *Let  $H$  a compact subgroup of  $G$  and  $X$  a metrizable proper  $G$ -space admitting a global  $H$ -slice  $S$ . Then there is a compatible  $G$ -invariant metric  $d$  on  $X$  such that each open unit ball  $O_d(x, 1)$  is a small set.*

**Lemma 3.2.** *Let  $H$  be a compact subgroup of  $G$ . Then a subset  $S \subset G/H$  is small if and only if the closure  $\overline{S}$  is compact.*

*Proof.* Assume that  $S$  is a small subset of  $G/H$ . Then there is a neighborhood  $U$  of the point  $eH \in G/H$  such that the transporter

$$\langle S, U \rangle = \{g \in G \mid gS \cap U \neq \emptyset\}$$

has a compact closure. Due to local compactness of  $G/H$  one can assume that the closure  $\overline{U}$  is compact.

Next, for every  $sH \in S$  we have  $s^{-1}sH = eH \in U$ , i.e.,  $s^{-1} \in \langle S, U \rangle$ , or equivalently,  $s \in \langle U, S \rangle$ . Hence  $sH \in \langle U, S \rangle(U)$  showing that  $S \subset \langle U, S \rangle(U) \subset \overline{\langle U, S \rangle}(\overline{U})$ . But, due to compactness of the closures  $\overline{\langle U, S \rangle}$  and  $\overline{U}$ , the set  $\overline{\langle U, S \rangle}(\overline{U})$  is compact. This yields that the closure  $\overline{S}$  is compact, as required.

The converse is immediate from the fact that  $G/H$  is a proper  $G$ -space and every compact subset of a proper  $G$ -space is a small set (see [31, §1.2]).

□

**Corollary 3.3.** *Let  $H$  be a compact subgroup of  $G$  such that the quotient  $G/H$  is metrizable. Then there is a compatible  $G$ -invariant metric  $\rho$  on  $G/H$  such that each closed unit ball  $B_\rho(x, 1)$ ,  $x \in G/H$ , is compact.*

*Proof.* By lemmas 3.1 and 3.2, there exists a compatible  $G$ -invariant metric  $d$  in  $G/H$  such that each open unit ball  $O_d(x, 1)$  has a compact closure. Then the metric  $\rho$  defined by  $\rho(x, y) = 2d(x, y)$ ,  $x, y \in G/H$ , has the desired property because  $B_\rho(x, 1) \subset \overline{O_d(x, 1)}$ . □

We recall that a continuous function  $f : X \rightarrow \mathbb{R}$  defined on a  $G$ -space  $X$  is called  $G$ -uniform if for each  $\epsilon > 0$ , there is an identity neighborhood  $U$  in  $G$  such that  $|f(gx) - f(x)| < \epsilon$  for all  $x \in X$  and  $g \in U$ .

For a proper  $G$ -space  $X$  we denote by  $\mathcal{P}(X)$  the linear space of all bounded  $G$ -uniform functions  $f : X \rightarrow \mathbb{R}$  whose support  $\text{supp } f = \{x \in X \mid f(x) \neq 0\}$  is a small subset of  $X$ . We endow  $\mathcal{P}(X)$  with the sup-norm and the following  $G$ -action:

$$(g, f) \mapsto gf, \quad (gf)(x) = f(g^{-1}x), \quad x \in X.$$

It is easy to see that  $\mathcal{P}(X)$  is a normed linear  $G$ -space. It was proved in [2, Proposition 3.1] that the complement  $\mathcal{P}_0(X) = \mathcal{P}(X) \setminus \{0\}$  is a proper  $G$ -space. Moreover, it follows immediately from [12, propositiones 3.4 and 3.5] that the complement  $\widetilde{\mathcal{P}}_0(X) = \widetilde{\mathcal{P}}(X) \setminus \{0\}$  is also a proper  $G$ -space, where  $\widetilde{\mathcal{P}}(X)$  denotes the completion of  $\mathcal{P}(X)$ .

The following result follows immediately from [2, Lemma 3.3] and [2, proof of Proposition 3.4]:

**Proposition 3.4.** *Let  $(X, \rho)$  be a metric proper  $G$ -space with an invariant metric  $\rho$  such that each closed unit ball in  $X$  is a small set. Then  $X$  admits a  $G$ -embedding  $i : X \hookrightarrow \mathcal{P}_0(X)$  such that:*

- (1)  $\|i(x) - i(y)\| \leq \rho(x, y)$  for all  $x, y \in X$ ,
- (2)  $\rho(x, y) = \|i(x) - i(y)\|$  whenever  $\rho(x, y) \leq 1$ ,
- (3)  $\|i(x) - i(y)\| \geq 1$  whenever  $\rho(x, y) > 1$ ,
- (4) the image  $i(X)$  is closed in its convex hull.

We aim at applying this result to the case  $X = G/H$ , where  $H$  is a compact subgroup of  $G$  such that  $G/H$  is metrizable.

As it follows from Lemma 3.2, in this specific case,  $\mathcal{P}(G/H)$  is just the space of all continuous functions  $G/H \rightarrow \mathbb{R}$  having a precompact support. Respectively,  $\widetilde{\mathcal{P}}(G/H)$  is the Banach space of all continuous functions  $G/H \rightarrow \mathbb{R}$  vanishing at infinity.

**Proposition 3.5.** *Let  $H$  be a compact subgroup of  $G$  such that the quotient space  $G/H$  is metrizable. Choose, by Corollary 3.3, a compactible  $G$ -invariant metric  $\rho$  on  $G/H$  such that each closed unit ball  $B_\rho(x, 1)$ ,  $x \in G/H$ , is compact. Then  $G/H$  admits a  $G$ -embedding  $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$  such that:*

- (a)  $\|i(x) - i(y)\| \leq \rho(x, y)$  for all  $x, y \in G/H$ ,
- (b)  $\rho(x, y) = \|i(x) - i(y)\|$  whenever  $\rho(x, y) \leq 1$ ,
- (c)  $\|i(x) - i(y)\| \geq 1$  whenever  $\rho(x, y) > 1$ ,
- (d) the image  $i(G/H)$  is closed in  $\widetilde{\mathcal{P}}(G/H)$ .

*Proof.* Since  $(G/H, \rho)$  satisfies the hypothesis of Proposition 3.4, there exists a topological  $G$ -embedding  $j : G/H \hookrightarrow \mathcal{P}_0(G/H)$  satisfying all the four properties in Proposition 3.4. Composing this  $G$ -embedding with the isometric  $G$ -embedding  $\mathcal{P}_0(G/H) \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$  we get a  $G$ -embedding  $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$  that also satisfies the four properties in Proposition 3.4. Hence, the above properties (a), (b) and (c) are fulfilled. So, only the last property (d) needs to be verified.

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $G/H$  such that  $(i(x_n))_{n \in \mathbb{N}}$  converges to a point  $f \in \widetilde{\mathcal{P}}(G/H)$ . One should check that  $f \in i(X)$ . Since  $(i(x_n))_{n \in \mathbb{N}}$  is a Cauchy sequence, it follows from the above property (b) that  $(x_n)_{n \in \mathbb{N}}$  is also Cauchy. Since each closed unit ball  $B_\rho(x, 1)$  is a compact subset of  $G/H$ , it then follows from [17, Ch. XIV, Th. 2.3] that  $\rho$  is a complete metric. Then  $(x_n)_{n \in \mathbb{N}}$  converges to a limit, say  $y \in G/H$ . By continuity of  $i$ , this implies that  $i(x_n) \rightsquigarrow i(y)$ , and hence,  $f = i(y) \in i(G/H)$ , as required.  $\square$

**Lemma 3.6.** *Let  $H$  be a compact subgroup of  $G$  and  $G_0$  the connected component of  $G$ . If  $G/H$  is locally connected then the subgroup  $G_0H \subset G$  is open and almost connected.*

*Proof.* Since the natural map

$$G/H \rightarrow G/G_0H, \quad gH \mapsto gG_0H$$

is open and the local connectedness is invariant under open maps, we infer that  $G/G_0H$  is locally connected. On the other hand

$$G/G_0H \cong \frac{G/G_0}{(G_0H)/G_0}.$$

Consequently,  $G/G_0H$ , being the quotient space of the totally disconnected group  $G/G_0$  is itself totally disconnected. Hence,  $G/G_0H$  should be discrete, implying that  $G_0H$  is an open subgroup of  $G$ .

To prove that  $G_0H$  is almost connected it suffices to observe that the quotient group  $G_0H/G_0$  is just the image of the compact group  $H$  under the natural homomorphism  $G \rightarrow G/G_0$ , and hence, is compact.  $\square$

**Proposition 3.7.** *Let  $H$  be a compact subgroup of  $G$  such that  $G/H$  is metrizable and locally connected. Then there exists a closed  $G$ -embedding  $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$  such that*

$$(3.1) \quad \begin{cases} \|i(x) - i(y)\| > 1/2 & \text{whenever } x \text{ and } y \text{ belong} \\ & \text{to different connected components of } G/H. \end{cases}$$

*Proof.* Since  $G/H$  is metrizable, by virtue of Corollary 3.3, there exists a compatible  $G$ -invariant metric  $d$  on  $G/H$  such that each closed unit ball  $B_d(x, 1)$ ,  $x \in G/H$ , is compact.

To shorten our notation, set  $S = G_0H/H$ , where  $G_0$  stands for the identity component of  $G$ . Since  $S$  is the image of  $G_0$  under the quotient map  $G \rightarrow G/H$  we infer that  $S$  is connected. On the other hand, it follows from Lemma 3.6 that  $S$  is an open and closed subset of  $G/H$ , so  $S$  should be a connected component of  $G/H$ . Further, it is easy to check that  $S$  is a global  $G_0H$ -slice for  $G/H$ . Thus,

$$G/H = G(S) = \bigsqcup_{g \in G} gS,$$

the disjoint union of closed and open connected components  $gS$ , one  $g$  out of every coset in  $G/G_0H$ .

Next we define a new metric  $\rho$  on  $G/H$  as follows:

$$\begin{cases} \text{if two points } x \text{ and } y \text{ of } G/H \text{ belong to the same connected component,} \\ \text{then we put } \rho(x, y) = d(x, y); \text{ otherwise we set } \rho(x, y) = d(x, y) + 1/2. \end{cases}$$

Clearly,  $\rho$  is a compatible metric for  $G/H$ . Since  $B_\rho(x, 1) \subset B_d(x, 1)$  for every  $x \in X$ , we see that each closed unit ball  $B_\rho(x, 1)$  is compact.

To see the  $G$ -invariance of  $\rho$  assume that  $x, y \in X$  and  $h \in G$ . If the points  $x$  and  $y$  are in the same connected component  $gS$  then the points  $hx$  and  $hy$  belong to the same connected component  $hgS$ . But then  $\rho(hx, hy) = d(hx, hy) = d(x, y) = \rho(x, y)$ , as required.

By the same argument, if  $x$  and  $y$  are in two different connected components, then  $hx$  and  $hy$  also belong to different connected components. In this case  $\rho(hx, hy) = d(hx, hy) + 1/2 = d(x, y) + 1/2 = \rho(x, y)$ , as required. Thus  $\rho$  is  $G$ -invariant.

Now, let  $i : G/H \hookrightarrow \widetilde{\mathcal{P}}_0(G/H)$  be the closed  $G$ -embedding from Proposition 3.5. It then follows from the properties (a) and (b) of Proposition 3.5 that (3.1) is satisfied, as required.  $\square$

#### 4. LARGE SUBGROUPS

Recall that the letter  $G$  always denotes a locally compact Hausdorff group unless otherwise stated.

**Definition 4.1.** *A compact subgroup  $H$  of  $G$  is called large if the quotient space  $G/H$  is locally connected and finite-dimensional.*

The notion of a large subgroup of a compact group first was singled out in 1991 by the author [6] in form of two other its characteristic properties, namely: “ $G/H$  is a manifold” and “ $G/H$  is a  $G$ -ANR”. More systematically this notion was studied later in [7] (for compact groups) and in [10] (for almost connected groups). In this section we shall investigate the remaining general case of an arbitrary locally compact group. Large subgroups play a central role also in section 5.

The following result is immediate from Lemma 3.6:

**Corollary 4.2.** *Let  $H$  be a large subgroup of  $G$  and  $G_0$  the connected component of  $G$ . Then the subgroup  $G_0H \subset G$  is open and almost connected.*

**Proposition 4.3.** *Let  $H$  and  $K$  be compact subgroups of  $G$  such that  $H \subset K$ . If  $H$  is a large subgroup then so is  $K$ .*

*Proof.* Being  $H$  a large subgroup of  $G$ , the quotient  $G/H$  is finite-dimensional and locally connected. Since the map  $G/H \rightarrow G/K$ ,  $gH \mapsto gK$ , is continuous and open, we infer that  $G/K$  is locally connected. Its finite-dimensionality follows from the one of  $G/H$  and the following equality (see [32, Theorem 10]):

$$(4.1) \quad \dim G/H = \dim G/K + \dim K/H.$$

$\square$

**Proposition 4.4.** *Let  $H$  and  $K$  be two compact subgroups of  $G$  such that  $K$  is a large subgroup of  $G$  while  $H$  is a large subgroup of  $K$ . Then  $H$  is a large subgroup of  $G$ .*

*Proof.* Being  $K$  a large subgroup of  $G$ , the quotient  $G/K$  is finite-dimensional and locally connected. Then the natural map  $G/H \rightarrow G/K$  is a locally trivial fibration with the fibers homeomorphic to  $K/H$  (see [32, Theorem 13']). But  $K/H$  is also locally connected



(and finite-dimensional) since  $H$  is a large subgroup of  $K$ . This yields that  $G/H$  is locally connected. Finite-dimensionality of  $G/H$  follows from the one of  $G/K$  and  $K/H$ , and the above equality (4.1).

Thus,  $G/H$  is locally connected and finite-dimensional, and hence,  $H$  is a large subgroup of  $G$ , as required.  $\square$

The following lemma is a well-known result in the theory of Lie groups (see e.g., [24, Ch. II, Theorem 4.2]):

**Lemma 4.5.** *Let  $\Gamma$  be a Lie group and  $\Delta$  a closed subgroup of  $\Gamma$ . Then the quotient space  $\Gamma/\Delta$  has a unique smooth structure with the property that the natural  $\Gamma$ -action on  $\Gamma/\Delta$  induced by left translations is smooth.*

The following theorem of Montgomery and Zippin (see [29, §6.2]) plays a key role in what follows:

**Theorem 4.6** (Montgomery-Zippin). *Let  $G$  be an almost connected group that acts effectively and transitively on a locally compact, locally connected, finite-dimensional space. Then  $G$  is a Lie group.*

**Lemma 4.7.** *Every almost connected group is  $\sigma$ -compact.*

*Proof.* Let  $G$  be an almost connected group. By a well-known Malcev-Iwasawa theorem (see [33, Ch. H, Theorem 32.5]),  $G$  has a maximal compact subgroup  $K$ , i.e., every compact subgroup of  $G$  is conjugate to a subgroup of  $K$ . According to [12, Theorem 8.5],  $G = KG_0$  where  $G_0$  denotes the connected component of  $G$ . On the other hand, it is a well-known easy fact that every connected locally compact group is  $\sigma$ -compact, so  $G_0 = \bigcup_{n=1}^{\infty} C_n$ , where  $C_1, C_2, \dots$ , are compact subsets of  $G_0$ . Then, clearly,  $G = KG_0 = \bigcup_{n=1}^{\infty} KC_n$ , and since  $KC_n$  is compact for every  $n = 1, 2, \dots$ , we conclude that  $G$  is  $\sigma$ -compact.  $\square$

The following version of the above Montgomery-Zippin theorem plays an important role in the proof of Proposition 4.11 below:

**Theorem 4.8.** *Let  $G$  be an almost connected group that acts effectively and transitively on a locally compact, locally contractible space. Then  $G$  is a Lie group.*

*Proof.* This theorem is essentially proved in J. Szente [34]: indeed, since by Lemma 4.7, every almost connected group is  $\sigma$ -compact, the assertion follows from [34, Theorem 4].  $\square$

**Proposition 4.9.** *Let  $H$  be a compact subgroup of an almost connected group  $G$ . Then the following assertions are equivalent:*

- (1)  $H$  is a large subgroup,

- (2) *There exists a compact normal subgroup  $N$  of  $G$  such that  $N \subset H$  and  $G/N$  is a Lie group. In particular,  $G/H$  is a coset space of a Lie group.*
- (3)  *$G/H$  is locally contractible.*

*Proof.* Denote by  $N$  the kernel of the  $G$ -action on  $G/H$ , i.e.,

$$N = \{g \in G \mid gt = t \text{ for all } t \in G/H\}.$$

Evidently,  $N \subset H$  and the group  $G/N$  acts effectively and transitively on  $G/H$ . Since  $G$  is an almost connected group, by virtue of [12, Proposition 8.1], the quotient group  $G/N$  is also almost connected.

(1)  $\implies$  (2). Since  $G/H$  is locally connected and finite-dimensional, one can apply Theorem 4.6 according to which  $G/N$  is a Lie group. Since  $H/N$  is a compact subgroup of  $G/N$ , by Lemma 4.5, the coset space  $\frac{G/N}{H/N}$  is a smooth manifold.

It remains to observe that the following homeomorphism holds:

$$(4.2) \quad G/H \cong \frac{G/N}{H/N}.$$

(2)  $\implies$  (3) is evident.

(3)  $\implies$  (1). As above, denote by  $N$  the kernel of the  $G$ -action on  $G/H$ .

Evidently,  $N \subset H$  and the group  $G/N$  acts effectively and transitively on  $G/H$ . Besides, since  $G$  is an almost connected group, by virtue of [12, Proposition 8.1], the quotient group  $G/N$  is also almost connected.

Consequently, one can apply Theorem 4.8 according to which  $G/N$  is a Lie group. Since  $H/N$  is a compact subgroup of  $G/N$ , by Lemma 4.5, the coset space  $\frac{G/N}{H/N}$  is a smooth manifold.

Due to the homeomorphism (4.2) we get that  $G/H$  is a manifold, and in particular, it is finite-dimensional and locally connected. Thus,  $H$  is a large subgroup of  $G$ .  $\square$

**Remark 4.10.** *In [10, Proposition 3.2] it is claimed that the assertions (2) and (3) in this proposition are equivalent even for arbitrary locally compact group  $G$ . Unfortunately, the proof given in [10] contains a gap; in fact it is valid only for almost connected  $G$ . The error occurred because in [10] Theorem 4.8 was inaccurately applied to arbitrary locally compact groups while it is stated only for almost connected ones.*

However, for arbitrary locally compact groups we have the following characterization of large subgroups:

**Proposition 4.11.** *Let  $H$  be a compact subgroup of  $G$ . Then the following conditions are equivalent:*

- (1)  *$H$  is a large subgroup,*
- (2)  *$G/H$  is a smooth manifold; in this case it is the disjoint union of open submanifolds which all are homeomorphic to the same coset space of a Lie group,*

(3)  $G/H$  is locally contractible.

*Proof.* (1)  $\implies$  (2). Since, by Corollary 4.2,  $G_0H$  is open in  $G$  we see that  $G$  is the disjoint union of the open cosets  $xG_0H$ ,  $x \in G$ . Since the quotient map  $p : G \rightarrow G/H$  is continuous, open and closed we infer that  $G_0H/H$  is open and closed in  $G/H$ , and  $G/H$  is the disjoint union of its open subsets  $xG_0H/H$ ,  $x \in G$ . Observe that each  $xG_0H/H$  is homeomorphic to the coset space  $G_0H/H$ .

Hence, by virtue of Proposition 4.9, it suffices to show that  $H$  is a large subgroup of the almost connected group  $G_0H$  (see Corollary 4.2).

But this is easy. Indeed, since  $G_0H/H$  is an open subset of the locally connected space  $G/H$ , we infer that  $G_0H/H$  is locally connected too. Further, since  $G_0H/H$  is closed in  $G/H$  we infer that  $\dim G_0H/H \leq \dim G/H$ , and hence,  $G_0H/H$  is finite-dimensional because  $G/H$  is so. Thus,  $H$  is a large subgroup of  $G_0H$ , as required.

(2)  $\implies$  (3) is evident.

(3)  $\implies$  (1). Since local contractibility yields local connectedness, it follows from Corollary 4.2 that  $G_0H$  is an open almost connected subgroup of  $G$ . In turn,  $G_0H/H$  is an open subset of the locally contractible space  $G/H$ , and hence, is itself locally contractible. Then we can apply Proposition 4.9 to the almost connected group  $G_0H$  according to which  $H$  is a large subgroup of  $G_0H$ , and hence, the quotient  $G_0H/H$  is a manifold.

But  $G/H$  is the disjoint union of its open and closed subsets all homeomorphic to  $G_0H/H$ . This yields that  $G/H$  is a manifold, and in particular, it is finite-dimensional and locally connected. Thus,  $H$  is a large subgroup of  $G$ .  $\square$

The following result is proved in Elfving [18, p. 23-24] in a different way:

**Proposition 4.12.** *Let  $G$  be a Lie group and  $H$  a compact subgroup of  $G$ . Then  $G/H$  is a  $G$ -ANE( $\mathcal{P}$ ).*

*Proof.* By virtue of Corollary 3.3, there exists a compatible  $G$ -invariant metric  $\rho$  on  $G/H$  such that each closed unit ball  $B_\rho(x, 1)$ ,  $x \in G/H$ , is compact.

Next, by Proposition 3.5, one can assume that  $G/H$  is an invariant closed subset of the proper  $G$ -space  $\widetilde{\mathcal{P}}_0(G/H)$ . Now, due to Exact slice theorem 2.5 (see also [31, Corollary 1]),  $G/H$  is a  $G$ -retract of some invariant neighborhood  $U$  in  $\widetilde{\mathcal{P}}_0(G/H)$ . Since by Corollary 2.3,  $\widetilde{\mathcal{P}}(G/H) \in G\text{-AE}(\mathcal{P})$ , we conclude that  $G/H \in G\text{-ANE}(\mathcal{P})$ , as required.  $\square$

**Proposition 4.13.** *Let  $G$  be an almost connected group and  $H$  a large subgroup of  $G$ . Then  $G/H$  is a metrizable  $G$ -ANE( $\mathcal{P}$ ).*

*Proof.* Denote by  $N$  the kernel of the  $G$ -action on  $G/H$ , i.e.,

$$N = \{g \in G \mid gt = t \text{ for all } t \in G/H\}.$$

Then  $N \subset H$  and the group  $G/N$  acts effectively and transitively on the finite-dimensional locally connected space  $G/H$ . Consequently, according to Theorem 4.6,  $G/N$  is a Lie group.

We have to return again to the  $G$ -equivariant homeomorphism (4.2).

Since  $G/N$  is a Lie group, it then follows from Proposition 4.12 and homeomorphism (4.2) that  $G/H$  is a  $G/N$ -ANE( $\mathcal{P}$ ). Further,  $G/H$  is metrizable since it is homeomorphic to the coset space of the metrizable (in fact Lie) group  $G/N$ .

Now, since  $N$  acts trivially on  $G/H$ , it then follows from [4, Proposition 3] that  $G/H$  is a  $G$ -ANE( $\mathcal{P}$ ).  $\square$

We have developed all the tools necessary to prove our main result:

**Theorem 4.14.** *Let  $H$  be a compact subgroup of  $G$ . Then the following conditions are equivalent:*

- (1)  $H$  is a large subgroup,
- (2)  $G/H$  is a metrizable  $G$ -ANE( $\mathcal{P}$ ),
- (3)  $G/H$  is an ANE( $\mathcal{P}$ ).

*Proof.* (1)  $\implies$  (2). By Proposition 4.11,  $G/H$  is metrizable. Then, by Proposition 3.7, one can assume that  $G/H$  is a  $G$ -invariant closed subset of  $\widetilde{\mathcal{P}}_0(G/H)$  and

$$(4.3) \quad \begin{cases} \|x - y\| > 1/2 & \text{whenever } x \text{ and } y \text{ belong} \\ & \text{to different connected components of } G/H. \end{cases}$$

Set  $S = G_0H/H$  and denote by  $W$  the  $1/4$ -neighborhood of  $S$  in  $\widetilde{\mathcal{P}}_0(G/H)$ , i.e.,

$$W = \{z \in \widetilde{\mathcal{P}}_0(G/H) \mid \text{dist}(z, S) < 1/4\}.$$

*Claim.*  $W$  is a  $G_0H$ -slice in  $\widetilde{\mathcal{P}}_0(G/H)$ .

Indeed,  $W$  is  $G_0H$ -invariant since  $S$  is so and the norm of  $\widetilde{\mathcal{P}}(G/H)$  is  $G$ -invariant. Further,  $G(W)$  is open in  $\widetilde{\mathcal{P}}_0(G/H)$  since  $W$  is so.

Check that  $W$  and  $gW$  are disjoint whenever  $g \in G \setminus G_0H$ . In fact, if  $gw \in W$  for some  $w \in W$  then  $\|w - s\| < 1/4$  and  $\|gw - s_1\| < 1/4$  for some  $s, s_1 \in S$ . By the invariance of the norm we have  $\|gw - gs\| = \|w - s\| < 1/4$ . Hence,

$$\|gs - s_1\| \leq \|gs - gw\| + \|gw - s_1\| < 1/4 + 1/4 = 1/2.$$

Consequently, by (4.3),  $s_1$  and  $gs$  must belong to the same connected component of  $G/H$ . Since  $s_1 \in S$  we infer that  $gs \in S$ . Thus,  $S \cap gS \neq \emptyset$ . But, since  $S$  is a global  $G_0H$ -slice of  $G/H$  (see the proof of Proposition 3.7), it then follows that  $g \in G_0H$ , as required.

Thus,  $G(W)$  is the disjoint union of its open subsets  $gW$ . In particular, each  $gW$  is also closed in  $G(W)$ .

So, we have verified that  $W$  is a  $G_0H$ -slice in  $\widetilde{\mathcal{P}}_0(G/H)$ .

Further, since  $G_0H$  is an almost connected group (see Corollary 4.2), it follows from Proposition 4.13 that  $S$  is a  $G_0H$ -ANE( $\mathcal{P}$ ). Now, since  $W \in G_0H\text{-}\mathcal{P}$ , it then follows that

there exists a  $G_0H$ -equivariant retraction  $r : V \rightarrow S$  for some  $G_0H$ -invariant neighborhood  $V$  of  $S$  in  $W$ .

Then  $r$  induces a  $G$ -map  $R : G(V) \rightarrow G/H$  by the rule:  $R(gv) = gr(v)$ , where  $g \in G$  and  $v \in V$  (see [16, Ch. I, Proposition 4.3]).

Clearly,  $R$  is a  $G$ -retraction, and hence,  $G/H$  being a  $G$ -neighborhood retract of the  $G$ -AE( $\mathcal{P}$ )-space  $\tilde{\mathcal{P}}(G/H)$  (see Corollary 2.3), is itself a  $G$ -ANE( $\mathcal{P}$ )-space.

(2)  $\implies$  (3). Suppose that  $X$  is a paracompact space,  $A$  a closed subset of  $X$ , and  $f : A \rightarrow G/H$  a continuous map. Consider the  $G$ -space  $G \times X$  endowed with the action of  $G$  defined by the rule:  $h(g, x) = (hg, x)$  for all  $(g, x) \in G \times X$  and  $h \in G$ . Then the map  $F : G \times A \rightarrow G/H$ , given by  $F(g, a) = gf(a)$ , is a continuous  $G$ -map.

Since  $G$  is a proper  $G$ -space, so is the product  $G \times X$ . Since the orbit space of  $G \times X$  is evidently homeomorphic to  $X$ , we conclude that  $G \times X \in G\mathcal{P}$ . Hence, by the hypothesis, there exist a  $G$ -neighborhood  $U$  of  $G \times A$  in  $G \times X$  and a  $G$ -map  $\tilde{F} : U \rightarrow G/H$  that extends  $F$ . Since  $\{e\} \times A \subset U$  we infer that  $\{e\} \times V \subset U$  for some neighborhood  $V$  of  $A$  in  $X$ . But  $U$  is a  $G$ -invariant set, implying that  $G \times V \subset U$ .

Next we define a map  $\tilde{f} : V \rightarrow G/H$  by putting  $\tilde{f}(v) = \tilde{F}(e, v)$ . Clearly,  $\tilde{f}$  is a continuous extension of  $f$ , as required.

(3)  $\implies$  (1). By Proposition 4.11, it suffices to show that  $G/H$  is locally contractible. Assume that  $U$  is a compact neighborhood of the point  $a = eH$  in  $G/H$ . Denote by  $A$  the closed subset  $U \times \{0\} \cup \{a\} \times I \cup U \times \{1\}$  of the product  $U \times I$ , where  $I = [0, 1]$ , and consider the continuous map  $f : A \rightarrow G/H$  defined by the rule:

$$f(u, 0) = u \text{ and } f(u, 1) = a \text{ if } u \in U \text{ and } f(a, t) = a \text{ for all } t \in I.$$

Since  $U \times I$  is paracompact (in fact, compact) and  $G/H \in \text{ANE}(\mathcal{P})$ , the map  $f$  extends to a continuous map  $F : V \rightarrow G/H$  over an open neighborhood  $V$  of  $A$  in  $U \times I$ .

Choose an open neighborhood  $W$  of  $a$  in  $U$  such that  $W \times I \subset V$ . Then the restriction  $F|_{W \times I} : W \times I \rightarrow U$  is a contraction of  $W$  in  $U$  to the point  $a$ , as required.  $\square$

We conclude this section with the following two corollaries, which were used in the proof of [11, Theorem 1.1]:

**Corollary 4.15** ([11], Lemma 2.5). *Let  $H$  be a large subgroup of  $G$ . Assume that  $A$  is a closed invariant subset of a proper  $G$ -space  $X \in G\mathcal{P}$ , and  $S$  is a global  $H$ -slice of  $A$ . Then there exists an  $H$ -slice  $\tilde{S}$  in  $X$  such that  $\tilde{S} \cap A = S$ .*

*Proof.* Let  $f : A \rightarrow G/H$  be a  $G$ -map with  $f^{-1}(eH) = S$ . By Theorem 4.14,  $G/H \in G\text{-ANE}(\mathcal{P})$ , and hence, there exists a  $G$ -extension  $F : U \rightarrow G/H$  over an invariant neighborhood  $U$  of  $A$  in  $X$ . It is easy to see that the preimage  $\tilde{S} = F^{-1}(eH)$  is the desired  $H$ -slice.  $\square$

**Corollary 4.16** ([11], Lemma 3.2). *Let  $H$  be a closed normal subgroup of  $G$ , and  $K$  a large subgroup of  $G$ . Then  $(KH)/H$  is a large subgroup of  $G/H$ .*

*Proof.* Since  $(KH)/H$  is the image of the compact subgroup  $K$  under the continuous homomorphism  $G \rightarrow G/H$  we see that it is a compact subgroup of  $G/H$ . Further, observe that the following homeomorphism (even  $G$ -equivariant) holds:

$$(4.4) \quad \frac{G/H}{(KH)/H} \cong G/KH.$$

Since  $K$  is a large subgroup and  $K \subset KH$ , it follows from Proposition 4.3 that  $KH$  is so. Hence, the coset space  $G/KH$  is finite-dimensional and locally connected. It remains to apply (4.4).  $\square$

## 5. APPROXIMATE SLICES FOR PROPER ACTIONS OF NON-LIE GROUPS

In [1] and [10] approximate versions of the Exact slice theorem 2.5 were established which are applicable also to proper actions of non-Lie groups.

In this section we shall prove the following new version of the Approximate slice theorem for proper actions of *arbitrary locally compact groups* which improves the one in [10, Theorem 3.6]:

**Theorem 5.1** (Approximate slice theorem). *Assume that  $X$  is a proper  $G$ -space,  $x \in X$  and  $O$  a neighborhood of  $x$ . Denote by  $\mathcal{N}(x, O)$  the set of all large subgroups  $H$  of  $G$  such that  $G_x \subset H$  and  $H(x) \subset O$ . Then:*

- (1)  $\mathcal{N}(x, O)$  is not empty.
- (2) For every  $K \in \mathcal{N}(x, O)$ , there exists a  $K$ -slice  $S$  with  $x \in S \subset O$ .

In the proof of this theorem we shall need the following lemma:

**Lemma 5.2.** *Let  $X$  be a proper  $G$ -space,  $H$  a compact subgroup of  $G$ , and  $S$  a global  $H$ -slice of  $X$ . Then the restriction  $f : G \times S \rightarrow X$  of the action is an open map.*

*Proof.* Let  $O$  be an open subset of  $G$  and  $U$  be an open subset of  $S$ . It suffices to show that the set  $OU = \{gu \mid g \in O, u \in U\}$  is open in  $X$ .

Define  $W = \bigcup_{h \in H} (Oh^{-1}) \times (hU)$ . We claim that

$$(5.1) \quad X \setminus OU = f((G \times S) \setminus W).$$

Indeed, since  $OU = f(W)$  and  $X = f(G \times S)$ , the inclusion  $X \setminus OU \subset f((G \times S) \setminus W)$  follows.

Let us establish the converse inclusion  $f((G \times S) \setminus W) \subset X \setminus OU$ .

Assume the contrary, that there exists a point  $gs \in f((G \times S) \setminus W)$  with  $(g, s) \in (G \times S) \setminus W$  such that  $gs \in OU$ . Then  $gs = tu$  for some  $(t, u) \in O \times U$ . Denote  $h = g^{-1}t$ . Then one has:

$$s = g^{-1}tu = hu \quad \text{and} \quad (g, s) = (t^{-1}g, g^{-1}tu) = (th^{-1}, hu) \in (Oh^{-1}) \times (hU).$$

Since both  $s$  and  $u$  belong to  $S$ , and  $s = hu$ , we conclude that  $h \in H$ . Consequently,  $(Oh^{-1}) \times (hU) \subset W$ , yielding that  $(g, s) \in W$ , a contradiction. Thus, the equality (5.1) is proved.

Being a global  $H$ -slice, the set  $S$  is a closed small subset of  $X$ . Consequently, by virtue of [1, Proposition 1.4], the restriction of the action map  $G \times S \rightarrow X$  is closed. Then, since  $(G \times S) \setminus W$  is a closed subset of  $G \times S$ , the image  $f((G \times S) \setminus W)$  is closed in  $X$ . Finally, together with the equality (5.1), this implies that  $OU$  is open in  $X$ , as required.  $\square$

*Proof of Theorem 5.1.* We first consider two special cases, namely  $G$  totally disconnected and  $G$  almost connected, and combine the two to get the general result.

Consider the set  $V = \{g \in G \mid gx \in O\}$  which is an open neighborhood of the compact subgroup  $G_x$  in  $G$ .

*Case 1.* Let  $G$  be totally disconnected. Then there exists a compact open subgroup  $H$  of  $G$  such that  $G_x \subset H \subset V$  (see [29, Ch. II, § 2.3]). Therefore  $G/H$  is discrete, and hence,  $H$  is a large subgroup of  $G$ . Thus,  $H \in \mathcal{N}(x, O)$ .

Now assume that  $K \in \mathcal{N}(x, O)$ . Then  $K$  is a compact open subgroup of  $G$  (see, e.g., Corollary 4.2). Since  $K(x) \subset O$ , there exists a neighborhood  $Q$  of  $x$  such that  $KQ \subset O$ . Since  $K$  is open, by [31, Proposition 1.1.6], there exists a neighborhood  $W$  of the point  $x$  in  $X$  such that  $\langle W, W \rangle \subset K$ . Then the set  $S = K(Q \cap W)$  is a  $K$ -invariant neighborhood of  $x$  with  $S \subset O$  and  $\langle S, S \rangle = K^{-1} \langle Q \cap W, Q \cap W \rangle K = K$ . Now, the saturation  $U = G(S)$  is the disjoint union of open subsets  $gS$ , one  $g$  out of every coset in  $G/K$ . So, the map  $f : U \rightarrow G/K$  with  $f(u) = gK$  if  $u \in gS$ , is a well-defined  $G$ -map and  $f^{-1}(eK) = S$ . Since  $x \in S \subset O$ , we are done.

*Case 2.* Let  $G$  be almost connected. By compactness of  $G_x$ , there exists a unity neighborhood  $V_1$  in  $G$  such that  $V_1 \cdot G_x \subset V$ . By a result of Yamabe (see [29, Ch. IV, § 46] or [21, Theorem 8]),  $V_1$  contains a compact normal subgroup  $N$  of  $G$  such that  $G/N$  is a Lie group; in particular,  $N$  is a large subgroup of  $G$ . Setting  $H = N \cdot G_x$  we get a compact subgroup  $H$  of  $G$  such that  $G_x \subset H \subset V$ . Since  $N \subset H$  and  $N$  is a large subgroup, it follows from Proposition 4.3 that  $H$  is also a large subgroup. Thus,  $H \in \mathcal{N}(x, O)$ .

Now assume that  $K \in \mathcal{N}(x, O)$ . Denote by  $M$  the kernel of the  $G$ -action on  $G/K$ , i.e.,

$$M = \{g \in G \mid gt = t \text{ for all } t \in G/K\}.$$

Then  $M \subset K$  is a compact normal subgroup of  $G$  and the group  $G/M$  acts effectively and transitively on the finite-dimensional locally connected space  $G/K$ . Consequently, according to Theorem 4.6,  $G/M$  is a Lie group.

Since  $K$  is compact and  $K(x) \subset O$ , there exists a  $K$ -invariant neighborhood  $Q$  of  $x$  such that  $Q \subset O$ . Let  $p : X \rightarrow X/M$  be the  $M$ -orbit map. Then  $X/M$  is a proper  $G/M$ -space [31, Proposition 1.3.2], and it is easy to see that the  $G/M$ -stabilizer of the point  $p(x) \in X/M$  is just the group  $K/M$ . Now, by the Exact slice theorem 2.5, there exists an invariant neighborhood  $\tilde{U}$  of  $p(x)$  in  $X/M$  and a  $G/M$ -equivariant map

$$\tilde{f} : \tilde{U} \rightarrow \frac{G/M}{K/M}$$

such that  $\tilde{f}(p(x)) = K/M$ .

Next we shall consider  $X/M$  (and its invariant subsets) as a  $G$ -space endowed with the action of  $G$  defined by the natural homomorphism  $G \rightarrow G/M$ . In particular,  $\tilde{U}$  is a  $G$ -space.

Since the two  $G$ -spaces  $\frac{G/M}{K/M}$  and  $G/K$  are naturally  $G$ -homeomorphic, we can consider  $\tilde{f}$  as a  $G$ -equivariant map from  $\tilde{U}$  to  $G/K$  with  $\tilde{f}(p(x)) = eK$ .

Let  $S = \tilde{f}^{-1}(eK)$  and  $S_1 = S \cap p(Q)$ . Then  $S$  is a global  $K$ -slice for  $\tilde{U}$  and  $S_1$  is an open  $K$ -invariant subset of  $S$ .

We claim that the  $G$ -saturation  $U_1 = G(S_1)$  is a tubular set with  $S_1$  as a  $K$ -slice. Indeed, the openness of  $U_1$  in  $\tilde{U}$ , and hence in  $X/M$ , follows from Lemma 5.2.

To prove that  $S_1$  is a global  $K$ -slice of  $U_1$  it suffices to show that  $f_1^{-1}(eK) = S_1$ , where  $f_1 : U_1 \rightarrow G/K$  is the restriction  $\tilde{f}|_{U_1}$ .

To this end, choose  $x \in f_1^{-1}(eK)$  arbitrary. Since  $f_1^{-1}(eK) = S \cap U_1$  then  $x = gs_1$  for some  $s_1 \in S_1$  and  $g \in G$ . Hence,  $gs_1 \in S \cap gS$ , which implies that  $g \in K$ . Since  $S_1$  is  $K$ -invariant we infer that  $x = gs_1 \in S_1$ . Thus,  $f_1^{-1}(eK) \subset S_1$ . The converse inclusion  $S_1 \subset f_1^{-1}(eK)$  is evident, so we get the desired equality  $f_1^{-1}(eK) = S_1$ .

Thus,  $S_1$  is a  $K$ -slice lying in  $p(Q)$  and containing the point  $p(x) \in X/M$ .

Now we set  $U = p^{-1}(U_1)$ ,  $S = p^{-1}(S_1)$ , and let  $f : U \rightarrow G/K$  be the composition  $f_1 p$ . Since  $S = f^{-1}(eK) \subset p^{-1}(p(Q)) = Q \subset O$  and  $x \in S$ , we conclude that  $S$  is the desired  $K$ -slice.

*Case 3.* Let  $G$  be arbitrary. First we show that  $\mathcal{N}(x, O) \neq \emptyset$ .

Denote by  $G_0$  the identity component of  $G$  and  $\tilde{G} = G/G_0$ . Set  $\tilde{X} = X/G_0$  and let  $p : X \rightarrow \tilde{X}$  be the  $G_0$ -orbit map. Then  $\tilde{X}$  is a proper  $\tilde{G}$ -space [31, Proposition 1.3.2], and the stabilizer  $\tilde{G}_{p(x)}$  of the point  $p(x) \in X/M$  in  $\tilde{G}$  is just the group  $(G_0 \cdot G_x)/G_0$ .

Since  $\tilde{G}$  is totally disconnected, there exists a compact open subgroup  $M$  of  $\tilde{G}$  such that  $\tilde{G}_{p(x)} \subset M$  (see [29, Ch. II, § 2.3]).

Denote by  $\pi : G \rightarrow \tilde{G}$  the natural homomorphism and let  $L = \pi^{-1}(M)$ . Then  $L$  is a closed-open subgroup of  $G$ . Since, clearly, the quotient group  $G/G_0$  is topologically isomorphic to the compact group  $M$  we infer that  $L$  is almost connected. Hence, we can apply the first part of case 2 to the almost connected group  $L$ , the proper  $L$ -space  $X$  and the neighborhood  $O \subset X$  of the point  $x \in X$ . Accordingly, there exists a large subgroup  $N$  of  $L$  such that  $L_x \subset N$  and  $N(x) \subset O$ .



We claim that  $N \in \mathcal{N}(x, O)$ . Indeed, since  $(G_0 \cdot G_x)/G_0 = \tilde{G}_{p(x)} \subset M$  we infer that  $G_0 \cdot G_x \subset L$ . In particular, this yields that  $G_x = L_x$ , and hence,  $G_x \subset N$ . It remains to check that  $N$  is a large subgroup of  $G$ . In fact, since  $N$  is a large subgroup of  $L$ , due to Proposition 4.11, the quotient  $L/N$  is locally contractible. But  $G/N$  is the disjoint union of its open subsets of the form  $xL/N$ ,  $x \in G$ , each of which is homeomorphic to  $L/N$ . Consequently,  $G/N$  is itself locally contractible, and again by Proposition 4.11, this yields that  $N$  is a large subgroup of  $G$ . Thus, we have proved that  $N \in \mathcal{N}(x, O)$ , as required.

Next, we assume that  $K \in \mathcal{N}(x, O)$ . Since  $K$  is a large subgroup of  $G$ , by Corollary 4.2,  $H = G_0K$  is an open almost connected subgroup of  $G$ . Hence  $\tilde{H} = G_0K/G_0$  is a compact open subgroup of  $\tilde{G}$ . The inclusion  $G_x \subset K$  easily implies that  $\tilde{G}_{p(x)} \subset \tilde{H}$ . Respectively, the inclusion  $K(x) \subset O$  yields that  $\tilde{H} \subset p(O)$ . Then, according to case 1, there exists a  $\tilde{G}$ -map  $f_1 : U_1 \rightarrow \tilde{G}/\tilde{H}$  of an open  $\tilde{G}$ -invariant neighborhood  $U_1$  of  $p(x)$  in  $\tilde{X}$  to the discrete  $\tilde{G}$ -space  $\tilde{G}/\tilde{H}$  with  $p(x) \in f_1^{-1}(\tilde{e}\tilde{H}) \subset p(O)$ .

The inverse image  $W_1 = f_1^{-1}(\tilde{e}\tilde{H})$  is an open  $\tilde{H}$ -invariant subset of  $\tilde{X}$ ; so the set  $W = p^{-1}(W_1)$  is an open  $H$ -invariant subset of  $X$  with  $x \in W$ ,  $G_x \subset K \subset H$  and  $K(x) \subset W \cap O$ . Since  $H/K$  is open in  $G/K$  we infer that  $K$  is a large subgroup of  $H$  (for instance, by Proposition 4.11).

Hence, we can and do apply case 2 of this proof to the almost connected group  $H$ , the proper  $H$ -space  $W$ , the neighborhood  $O \cap W$  of the point  $x \in W$  and the large subgroup  $K$  of  $H$ . Then there exist an  $H$ -neighborhood  $U$  of  $x$  in  $W$  and an  $H$ -map  $f_0 : U \rightarrow H/K$  with  $x \in f_0^{-1}(eK) \subset O \cap W$ . Next we want to extend  $f_0$  to a  $G$ -map  $f : G(U) \rightarrow G/K$ . Since  $H/K \subset G/K$ , we simply define  $f(gu) = gf_0(u)$  for  $g \in G$ ,  $u \in U$ .

It is easy to check that  $f$  is a well-defined  $G$ -map. Thus, the  $K$ -slice  $S = f^{-1}(eK)$  is the desired one.  $\square$

If  $G$  is a Lie group then, clearly, each compact subgroup of  $G$  is large. So, in this case, Theorem 5.1 has the following simpler form:

**Corollary 5.3.** *Assume that  $G$  is a Lie group,  $X$  a proper  $G$ -space,  $x \in X$  and  $O$  a neighborhood of  $x$ . Then for each compact subgroups  $K$  of  $G$  such that  $G_x \subset K$  and  $K(x) \subset O$ , there exists a  $K$ -slice  $S$  such that  $x \in S \subset O$ .*

We derive from Theorem 5.1 yet another corollary applicable to the, so-called, *rich*  $G$ -spaces.

Recall that a  $G$ -space  $X$  is called *rich*, if for any point  $x \in X$  and any its neighborhood  $U \subset X$ , there exists a point  $y \in U$  such that the stabilizer  $G_y$  is a large subgroup of  $G$  and  $G_x \subset G_y$  (see [7], [9]).

**Corollary 5.4.** *Assume that  $X$  is a rich proper  $G$ -space,  $x \in X$  and  $O$  a neighborhood of  $x$ . Then there exist a point  $y \in O$  with a large stabilizer  $G_y$  containing  $G_x$ , and a  $G_y$ -slice  $S$  such that  $x \in S \subset O$ .*

*Proof.* Choose a neighborhood  $O'$  of  $x$  such that  $G_y(x)$  is contained in  $O$  for all  $y \in O'$  (see [10, Lemma 3.9]). Further, since  $X$  is a rich proper  $G$ -space, we can choose a point  $y \in O'$  such that  $G_y$  is a large subgroup of  $G$  and  $G_x \subset G_y$ . Thus,  $G_y \in \mathcal{N}(x, O)$ , the set defined in Theorem 5.1.

If  $G(y) = G(x)$  then the stabilizer  $G_x$ , being conjugate to  $G_y$ , is also a large subgroup, and clearly,  $G_x \in \mathcal{N}(x, O)$ . Next, we apply item (2) of Theorem 5.1 to  $K = G_x$ ; the resulting  $K$ -slice  $S$  is the desired one.

If  $G(y) \neq G(x)$  then we first choose (due to [31, Proposition 1.2.8]) disjoint invariant neighborhoods  $A_x$  and  $A_y$  of  $G(x)$  and  $G(y)$ , respectively. Next, we apply twice the first assertion of the statement (2) of Theorem 5.1: first, to  $x \in O \cap A_x$  and  $K = G_y$ , and then to  $y \in O \cap A_y$  and  $K = G_y$ . As a result we get two  $G_y$ -slices  $S_x \subset O \cap A_x$  and  $S_y \subset O \cap A_y$  which contain the points  $x$  and  $y$ , respectively. Since  $A_x \cap A_y = \emptyset$  the union  $S = S_x \cup S_y$  is the desired  $G_y$ -slice.  $\square$

In conclusion we show that there are sufficiently many rich  $G$ -spaces. Indeed, it was proved in [7] that if  $G$  is a compact group then every metrizable  $G$ -ANE( $\mathcal{M}$ ) is a rich  $G$ -space, where  $G$ - $\mathcal{M}$  stands for the class of all proper  $G$ -spaces that are metrizable by a  $G$ -invariant metric. The same was proved in [10, Proposition 3.10] for proper actions of almost connected groups. Below we show that it is true also for proper actions of arbitrary locally compact groups.

**Proposition 5.5.** *Every metrizable proper  $G$ -ANE( $\mathcal{M}$ ) is a rich  $G$ -space.*

*Proof.* Let  $X$  be a metrizable proper  $G$ -ANE( $\mathcal{M}$ ),  $x \in X$  and  $O$  a neighborhood of  $x$ . Then by Theorem 5.1, there is a compact large subgroup  $K \subset G$ , and a  $K$ -slice  $S$  such that  $x \in S \subset O$ . The tube  $G(S)$ , being an open subset of  $X$ , is a  $G$ -ANE( $\mathcal{M}$ ) as well. This yields that  $G(S)$  is a  $K$ -ANE( $\mathcal{M}$ ) (see [10, Proposition 3.4]). Hence, according to [7, Proposition (2)],  $G(S)$  is a rich  $K$ -space, so there exists a point  $y \in G(S) \cap O$  such that  $K_x \subset K_y$ , and  $K_y$  is a large subgroup of  $K$ . It remains to show that  $G_y$  is a large subgroup of  $G$ .

First, it follows from Proposition 4.4 that  $K_y$  is a large subgroup of  $G$ . Next, since the point  $y$  belongs to the  $K$ -slice  $S$  we infer that  $K_y = G_y$ , and therefore,  $G_y$  is a large subgroup of  $G$ , as required.  $\square$

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